Notes on Rajan Part 2
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In discussion of the competition between an incumbent bank and a challenger bank, Rajan imagines an auction taking place. However the discussion of this auction is needlessly complicated. These notes will attempt to simplify and clarify. For any function $x(P)$ on the reals we will define

$$x_-(P) = \lim_{P^- \uparrow P} x(P^-) \text{ and } x_+(P) = \lim_{P^+ \downarrow P} x(P^+)$$

The environment is as follows. Risk neutrality, no discounting. The project is either successful or unsuccessful; both the borrower and the incumbent bank know which of the two possibilities has arisen. The challenger bank does not know, only having a belief that the probability of a successful project is q. If the project is successful, it will require an investment of a fixed amount I today, in return for which it will pay off a fixed amount X tomorrow. If the project is not successful, it will pay off 0 tomorrow; for simplicity any funds the borrower receives today (necessarily from the uninformed challenger bank) are simply squandered by the borrower.

Each of the two banks can make an offer to supply the amount I today in return for a payment tomorrow. If two offers are made, the borrower will accept the one with the lower payment. If the payment is less than X and the project is successful, the borrower will honor it; otherwise the bank will receive nothing. Clearly then banks will not make offers with payment terms outside the range $[0, X]$. (Note that we assume that if the best terms offered are $P = X$, so that the successful borrower is indifferent between accepting and not accepting, then the borrower accepts. Not clear that there would be any equilibrium under the alternative assumption). Assume $qX > I$; otherwise the challenger would never have a reason to participate.

Each bank chooses a distribution of offer terms where the distribution is $F(P)$ for the incumbent bank and $G(P)$ for the challenger bank. These distributions are chosen from the set of non-decreasing functions from $[0, X]$ to $[0, 1]$. For the challenger bank, the distribution represents the random distribution of terms offered to the borrower. We do not require $G_+(X)$ to equal 1; a value less than 1 represents the probability that no offer is made. For the incumbent bank the distribution is in principle contingent on whether the project is observed to be successful or not, but clearly, the incumbent will make no offer if he observes an unsuccessful project; thus $F(P)$ represents the random distribution of bids offered contingent on the project being observed to be successful. We are looking for a Nash equilibrium in the choices F and G.

Define for any non decreasing function H

$$\pi^H(P) = [1 - H_+(P) + \frac{1}{2}(H_+(P) - H_-(P))](P - I).$$

If the incumbent bank observes a successful project and offers terms $P \leq X$ his expected profits are $\pi^G(P)$. The complication of the notation is designed to handle ties. In other words, if the other bank's bid is greater than P then the incumbent will win the auction outright and this happens with probability $1-G_+(P)$. If the two bids are equal, then each bidder wins with equal probability. As we will see the probability of ties will turn out to be zero in equilibrium. If there is no atom in the distribution G at P; that is if $G_+(P) = G_-(P)$ the formula simplifies to

$$\pi^G(P) = [1 - G(P)](P - I).$$

Note that the incumbent bank's expected profits depend on G, the distribution of the challenger bank's bids.

If the challenger bank offers terms $P \leq X$ his expected profits are

$$q[1 - F_+(P) + \frac{1}{2}(F_+(P) - F_-(P))](P - I) - (1 - q)I = q\pi^F(P) - (1 - q)I$$

in other words, it is as if the two banks have symmetric payoff functions $\pi^G(P)$ and $\pi^F(P)$ except that the challenger bank pays a fixed cost of $I(1 - q)/q$ for bidding.

Clearly neither bidder will offer a P below I. Moreover, the challenger will never make a bid P below I/q because then

$$\begin{aligned} & q[1 - F_+(P) + \frac{1}{2}(F_+(P) - F_-(P))](P - I) - (1 - q)I \\ \leq \quad & q(P - I) - (1 - q)I < 0. \end{aligned}$$

But if it is certain that the challenger will not bid below I/q, then the incumbent will not bid below I/q; for any such bid strictly below I/q there is always another more profitable bid (for example half way between the proposed bid and I/q).

Our investigation is greatly simplified by the following lemmas:

Lemma 1 *In equilibrium,* $F_+(P) = F_-(P)$ *and* $G_+(P) = G_-(P)$ *for* $P < X$.
Proof. *Suppose that in equilibrium* $F_+(P) > F_-(P)$. *Then* $\pi_-^F(P) > \pi^F(P) > \pi_+^F(P)$, *so that there is an interval* $[P, P+\varepsilon]$ *in which the challenger will not bid. But this means that* $G_-(P) = G_+(P) = G_-(P+\varepsilon) = G_+(P+\varepsilon)$, *implying that* $\pi^G(P+\varepsilon) > \pi^G(P)$. *In other words, the incumbent's profits are greater from a bid of* $P+\varepsilon$, *contradicting the assumption that there was a positive probability of a bid by the incumbent at* P.

The argument is symmetric if we start from the assumption $G_+(P) > G_-(P)$. ∎

In a similar vein we can show that the support for the two distributions is the same: if there is an interval in which one of the banks makes no bids, the other bank has no incentive to make a bid in that interval either–any proposal for a bid in that interval is dominated by a bid at the upper end of the interval. Moreover there cannot be any gaps in the bidding:

Lemma 2 *Suppose* $P_1 < P_2 < X$.*In equilibrium, if* $F(P_1) > 0$, *then* $F(P_2) > F(P_1)$ *and similarly for* G.
Proof. *Suppose* $F(P_2) = F(P_1) > 0$. *Without loss of generality, let* P_1 *be the infimum of values of* P *for which* $F(P) = F(P_2)$. *That is, the incumbent has zero probability of bidding in the interval* $[P_1, P_2]$ *and therefore so does the challenger. But then bids in an interval* $[P_1 - \varepsilon, P_1]$ *are strictly dominated by bids of* P_2, *violating the assumption that bids are occuring in this interval with positive probability.* ∎

Thus we can simplify the search to non decreasing functions F and G which are continuous except possibly at X, and for which $F(I/q) = G(I/q) = 0$.

Let $\bar{\pi}^H$ denote

$$\sup_{P \preceq X} [1 - H(P)](P - I).$$

and let S^H denote the set of P where the supremum is attained. By what has been shown so far, $S^F = S^G = [I^*, X]$ for some $I^* \geq I/q$.

More concretely

$$\begin{aligned} 1 - \frac{\bar{\pi}^G}{P - I} &= G(P) \\ 1 - \frac{\bar{\pi}^F}{P - I} &= F(P) \end{aligned}$$

on the interval $[I^*, X]$, and we are simply looking for equilibria within set of values $I^*, \bar{\pi}^G, \bar{\pi}^F, F_-(X), G_-(X), F_+(X), G_+(X)$, where the first five parameters satisfy the following conditions:

$$\begin{aligned} I/q &\leq I^* \leq X \\ 1 - \frac{\bar{\pi}^G}{I^* - I} &= 0 = 1 - \frac{\bar{\pi}^F}{I^* - I} \\ F_-(X) &= G_-(X) = 1 - \frac{\bar{\pi}}{X - I} \end{aligned}$$

where $\bar{\pi}$ denotes the common value. This leaves us a one-parameter family to search. The final conditions to determine the equilibrium have to do with the decisions to bid the highest possible value X or not to bid at all. For convenience, let g denote $G_+(X) - G_-(X)$ and let f denote $F_+(X) - F_-(X)$). Then

$$\pi^G(X) = [1 - G_-(X) - \frac{1}{2}g](X - I)$$

and similarly for $\pi^F(X)$. It boils down to this:

1. Can there be an equilibrium in which the banks never bid below X? Suppose that the only two strategies chosen are bid X or don't bid. Let incumbent bid X with probability p_i; and let challenger bid with probability p_c. Incumbent's profit from bidding is

$$p_c(\frac{1}{2})(X-I)+(1-p_c)(X-I)$$

so incumbent will always bid. Challenger's profit from bidding X is then

$$q\frac{1}{2}(X-I)-(1-q)I.$$

Challenger's profit from bidding X^- is larger:

$$q(X-I)-(1-q)I.$$

Thus as long as $qX > I$, there is no equilibrium in which neither bank bids below X, and so, from before, in equilibrium both banks must bid below X with positive probability.

2. Can there be an equilibrium in which both banks bid X with positive probability? No, by arguments similar to those before, this would lead to an incentive for each to undercut the other; thus at most one can bid X with positive probability.

3. Can there be an equilibrium in which neither bank bids X with positive probability? Since $F_-(X) = G_-(X)$, there are only two possibilities to consider; either $F_-(X) = G_-(X) = 1$—that is, both banks bid less than X with certainty, or $F_-(X) = G_-(X) < 1$ and both banks don't bid with equal positive probability. In the first case $\bar{\pi} = 0$, so that the incumbent makes zero expected profits while the challenger makes negative expective profits $q0-(1-q)I$. So this is not an equilibrium. In the second case, the incumbent will make positive expected profits. But not bidding gives zero expected profits, so bidding must be giving positive expected profits. So it cannot be an equilibrium to mix between bidding and not bidding.

4. Thus one bank must bid X with positive probability and less than X with positive probability. The other must bid less than X with the same positive probability; must bid X with zero probability and so must refrain from bidding with positive probability. In order to refrain from bidding with positive probability that bank must be obtaining zero profits from bidding. Because of the information advantage the incumbent must be making positive profits, so the zero profits must be going to the challenger. With positive profits from bidding it cannot be an equilibrium for the incumbent to refrain from bidding.

We have almost pinned down the equilibrium. For a particular level of I^*, we will generate a positive probability H of bids below X. With probability $1-H$, the incumbent will bid X. With probability $1-H$, the challenger will not bid. (Recall of course that the incumbent's bids are actually contingent on observing a successful signal; the challenger's bids are uncontingent). The incumbent makes positive profits; the challenger makes zero profits. It is this last condition that pins down the value of I^*

$$q[1-F(I^*)](I^*-I)-(1-q)I=0$$

or

$$q(I^*-I)-(1-q)I$$

or

$$I^* = I/q$$

and so the profit of the incumbent is

$$\bar{\pi} = I^* - I = I\frac{1-q}{q}$$

the distribution of bids below X is determined as follows

$$\begin{aligned} F(P) &= G(P) = 0 \text{ for } P \leq I/q \\ F(P) &= G(P) = 1 - \frac{I}{P-I}\frac{1-q}{q} \text{ for } P \in [I/q, X) \end{aligned}$$

and with probability

$$1-F(X) = 1-G(X)$$

the incumbent firm bids X and the challenger does not bid.